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# APPROXIMATION MODEL OF A TYPICAL MECHANICAL OBJECT WITH DISTRIBUTED PARAMETERS

A.F. Verlan
*Pukhov Institute for Modelling in Energy Engineering*

M.V. Sagatov
*Tashkent State Technical University*, informtgtu@mail.ru

D.K. Karimova
*Tashkent State Technical University*

U.S. Fayzullaev
*Tashkent State Technical University*

# APPROXIMATION MODEL OF A TYPICAL MECHANICAL OBJECT WITH DISTRIBUTED PARAMETERS

**A.F. Verlan (Pukhov Institute for Modelling in Energy Engineering), M.V. Sagatov, D.K. Karimova, U.S. Fayzullaev (Tashkent State Technical University)**

***Annotation.*** The article discusses the method of approximation transformations for the study of a typical mechanical object with distributed parameters. A mathematical description of an object with distributed parameters, which is given in the form of a partial differential equation, and their structural models are considered. An approximation model has been obtained with a number of unique properties that have proven useful in the construction of structural models of electromechanical systems.

***Keywords:*** *approximation, mathematical model, electromechanical systems, partial differential equations*

**Introduction.** In connection with the increasing complexity of the dynamic systems under study, and hence the complexity of their models, the methods of nonequivalent (approximate) transformations of mathematical descriptions of systems [1] become significant. The basis of these methods is the ability to simplify the initial dynamic model by identifying and eliminating those components that weakly (based on some criterion) affect the simulation result [2].

A typical mechanical distributed link of many actuators of electromechanical systems is a rod that experiences various types of deformation. For the mathematical description of its dynamic properties, as a rule, a model is used in the form of a partial differential equation. In computer modeling of distributed objects, the mathematical models of which are represented in the form of partial differential equations, it becomes necessary to reduce them to a form that allows the use of standard operating units in modeling software. Since the distributed object is infinite-dimensional, it is possible to describe it with a finite-dimensional approximation model by discretizing the original equation by the spatial coordinate.

Consider a mathematical description of an object with distributed parameters, which is given in a rectangular region G, with a limit $G\{\alpha < x < \beta; \ y_0 < y < y_0 + l\}$ in the form of a partial differential equation

$$a(x,t)\frac{\partial^2 u}{\partial t^2} - b(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)\frac{\partial u}{\partial t} + d(x,t)u = f(x,t), \qquad a,b > 0 \quad (1)$$

with boundary conditions

$$\begin{cases} u(x_0,t) = \varphi_0(t); & u(x_0+l,t) = \varphi_l(t); & (t_0 \le t \le T), \\ u(x,t_0) = \psi_0(x); & u(x,T) = \psi_T(x); & (x_0 \le x \le x_0 + l), \end{cases} \qquad (2)$$

where $\varphi_0(t)$, $\varphi_l(t)$, $\psi_0(x)$, $\psi_T(x)$ — predefined functions.

Applying the direct method to equations (1), (2) we obtain a system of n ordinary second-order linear differential equations

$$a_k(t)\ddot{u}_k(t) - \frac{b_k(t)}{h^2}[u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)] + c_k(t)\dot{u}_k(t) +$$

$$+ d_k(t)u_k(t) = f_k(t) + O(h^2), \qquad (k=1,2,...,n), \qquad (3)$$

where$x_k = x_0 + kh$,  $(k = 0, 1, 2, ..., n)$, $h = \dfrac{l}{n+1}$,  $x_k(t) = u(t, x_k)$.

Neglecting the terms $O(h^2)$ in (3) and denoting by $U_k(t)$ the approximate values of the solution $u(t, x)$ on the line $x = x_k$ to determine them, we obtain the system of equations

$$a_k(t)\ddot{U}_k(t) - \frac{b_k(t)}{h^2}[U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)] + c_k(t)\dot{U}_k(t) + \tag{4}$$

$$+d_k(t)U_k(t) = f_k(t), \qquad (k = 1, 2, ..., n).$$

Using the boundary conditions on G, we have:

$$\left.\begin{array}{ll} U_0(t) = \varphi_0(t), & (\alpha \leq t \leq T); \\ U_{n+1}(t) = \varphi_l(t), & (\alpha \leq t \leq T); \\ U_k(t_0) = \psi_0(x_k), & U_k(T) = \psi_T(x_k); \quad (k = 1, 2, ..., n). \end{array}\right\} \tag{5}$$

So, the obtained model in the form of a system of differential equations (4) with conditions (5) approximates up to $O(h^2)$ differential equation (1) with boundary conditions (2). It should be noted that using the direct method, in fact, the initial model is decomposed into n structural elements, each of which implements a second-order differential equation. For the numerical implementation of the obtained approximation model, the simulink model shown in Fig.1.
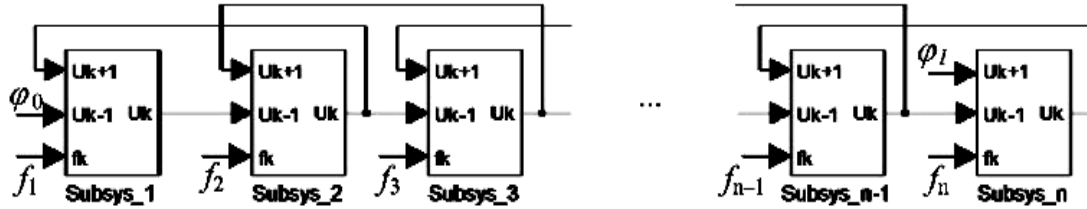


Fig.1. Structural implementation of the approximation model (4)

Thus, based on the performed decomposition, structural models of objects with distributed parameters are constructed. It should be borne in mind that replacing a model with an infinite number of degrees of freedom with a finite-dimensional one leads to differences in the values of their natural frequencies. To assess the accuracy of the approximation model, we use the definition of the difference in the natural frequencies of the original and approximation models.

Let us estimate the accuracy of the approximation when replacing a partial differential equation with a system of second-order differential equations, which corresponds to replacing a homogeneous linear extended object with longitudinal deformation by a multi-mass system. The natural frequencies of a homogeneous distributed object are found by solving the following problem:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0; \; \frac{\partial u}{\partial x}\bigg|_{x=0} = \frac{\partial u}{\partial x}\bigg|_{x=l} = 0,$$

where$u$ – is the offset of the cross section of the distributed object; $a^2 = \dfrac{Es}{\rho} = \dfrac{Esl}{M_0}$; $M_0$

— total mass of the distributed object.

The frequencies are found from the ratio

$$\omega_q = \frac{\pi a q}{l} = \pi \chi q, \quad q = 1, 2, ..., \tag{6}$$

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where $q$ – is the harmonic number; $\chi = \sqrt{\dfrac{k_0}{M_0}}$ ; $k_0 = \dfrac{Es}{l}$ — total stiffness of the distributed object.

As an estimate of the approximation accuracy, we take the magnitude of the difference in the natural frequencies of equivalent objects with distributed and concentrated parameters. To move to a multi-mass model, we divide the object into n identical sections. The mass of each section $m_i = M_0/n$ will be concentrated in its middle in the form of an absolutely rigid body. The stiffness between adjacent sections is taken equal to the stiffness of sections $k_{i,i+1} = nk_0$, $i = 1,2,\ldots,n-1$. In the future, to simplify the notation, we omit the indices for $k_{i,i+1}$ and $m_i$

To determine the eigenfrequencies of a multi-mass system, we denote the relative displacement of neighboring masses by $u_i$. Then, taking into account (4), we can write $n - 1$ equations

$$\begin{cases} mu_1'' + 2ku_1 - ku_2 = 0; \\ mu_2'' - ku_1 + 2ku_2 - ku_3 = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ mu_i'' - ku_{i-1} + 2ku_i - ku_{i+1} = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ mu_{n-1}'' - ku_{n-2} + 2ku_{n-1} = 0. \end{cases}$$

We write the characteristic equation of this system

$$\Delta_{n-1}(\omega) = k^{n-1} \times$$

$$\times \begin{bmatrix} 2 - \dfrac{m\omega^2}{k} & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 - \dfrac{m\omega^2}{k} & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 - \dfrac{m\omega^2}{k} & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 - \dfrac{m\omega^2}{k} \end{bmatrix} = \tag{7}$$

$$= k^{n-1}\Delta_{n-1}(\omega) = 0.$$

Disclosing the determinant $\Delta_{n-1}(\omega)$ in powers of $\omega^2$ and finding the eigenfrequencies requires even a relatively small order of the determinant significant computational work. To reduce the calculations and find the natural frequencies without solving the equation of the $(n - 1)$th order with respect to $\omega^2$, we can use the method of generating functions.

Let be the $G(z) = \displaystyle\sum_{n=0}^{\infty} \tilde{\Delta}_n z^n$ generating function of the sequence $\{\tilde{\Delta}_n\}$.

Opening the determinant of the elements of the first row and the algebraic complement to the term $-1$ in the first row of the elements of the first column, we obtain

$$\tilde{\Delta}_{i+2} = \left(2 - \dfrac{m\omega^2}{k}\right)\tilde{\Delta}_{i+1} - \tilde{\Delta}_i. \tag{8}$$

Wherein

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$$\tilde{\Delta}_0 = 1; \ \tilde{\Delta}_1 = 2 - \frac{m\omega^2}{k}. \tag{9}$$

Equation (8) is multiplied by $z^{i+2}$ and summed over $i$:

$$\sum_{i=0}^{\infty} \tilde{\Delta}_{i+2} z^{i+2} = \left(2 - \frac{m\omega^2}{k}\right) z \sum_{i=0}^{\infty} \Delta_{i+1} z^{i+1} - z^2 \sum_{i=0}^{\infty} \tilde{\Delta}_i z^i. \tag{10}$$

since
$$\sum_{i=0}^{\infty} \tilde{\Delta}_{i+2} z^{i+2} = \sum_{m=2}^{\infty} \tilde{\Delta}_m z^m = G(z) - \tilde{\Delta}_0 - \tilde{\Delta}_1 z;$$

$$\sum_{i=0}^{\infty} \tilde{\Delta}_{i+1} z^{k+2} = \sum_{m=1}^{\infty} \tilde{\Delta}_m z^m = G(z) - \tilde{\Delta}_0,$$

then equation (10) can be written as follows:

$$G(z) - \tilde{\Delta}_0 - \tilde{\Delta}_1 z = \left(2 - \frac{m\omega^2}{k}\right) z \left[G(z) - \tilde{\Delta}_0\right] - z^2 G(z).$$

Solving this equation with respect to $G(z)$ and using (9), we find the generating function

$$G(z) = \frac{1}{1 - \left(2 - \dfrac{m\omega^2}{k}\right) z + z^2}.$$

Since in the characteristic equation the diagonal terms are always positive, denoting

$$1 - \frac{m\omega^2}{k} = \cos\alpha, \tag{11}$$

we get
$$G(z) = \frac{1}{1 - \cos\alpha z + z^2}.$$

We develop the last expression for the generating function in a power series in $z$:

$$G(z) = \sum_{n=0}^{\infty} \frac{\sin\left[(n+1)\alpha\right]}{\sin\alpha} z^n. \tag{12}$$

The coefficients of this expansion are the desired $\tilde{\Delta}_n$. Equation (7) for natural frequencies in the case of a mechanical multi-mass system according to (12) will have the form $\tilde{\Delta}_{n-1}(\omega) = \sin n\alpha = 0$, whence $\alpha = \dfrac{\pi q}{n}$, $q = 1, 2, \ldots, n-1$. Taking into account (11), the natural frequencies of a system consisting of n masses are

$$\omega_q^{(n)} = 2n\chi \sin\frac{\pi q}{(2n)}.$$

With an increase in the number of concentrated masses, the low eigenfrequencies of a multi-mass system with lumped parameters will approach the corresponding values of the eigenfrequencies of an object with distributed parameters:

$$\lim_{\substack{n\to\infty \\ q\ll n}} \omega_q^{(n)} = \lim_{\substack{n\to\infty \\ q\ll n}} 2n\chi \sin\frac{\pi q}{2n} = \pi\chi q = \omega_q$$

The relative error of replacing an object with distributed parameters by a system with lumped parameters is

$$\delta_q^{(n)} = \frac{\omega_q - \omega_q^{(n)}}{\omega_q} = 1 - \frac{\sin\dfrac{\pi q}{2n}}{\dfrac{\pi q}{2n}}.$$ 

<div align="right">(13)</div>

It follows that the error does not depend on the parameters of the object, but is determined only by the number of concentrated masses $n$ and the harmonic number $q$. In fig. 2. The dependence of the relative error for various values of the harmonic number and the number of lumped masses when approximating by a multimass model is presented.
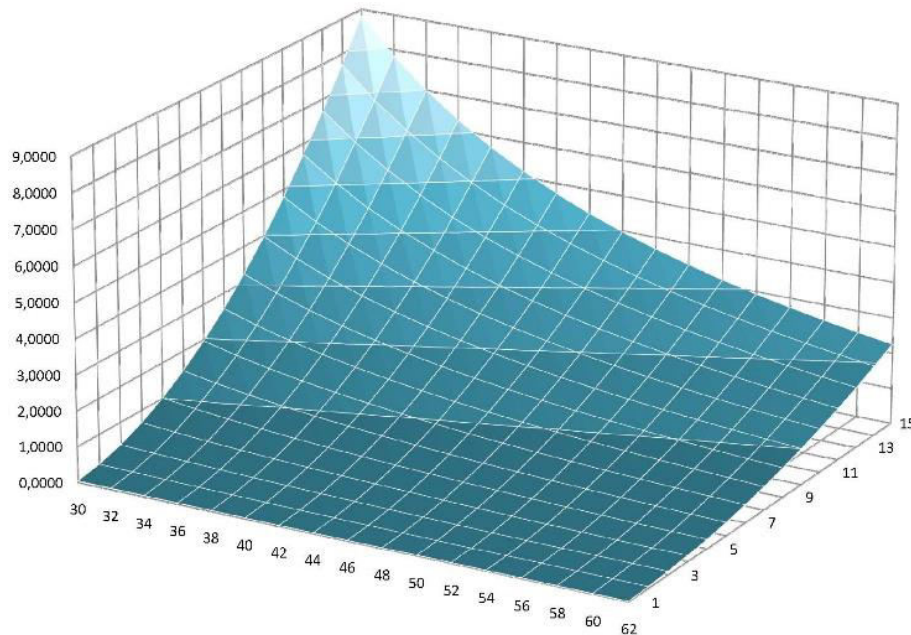


Fig. 2. The dependence of the relative error on the harmonic number and the number of concentrated masses

Thus, asking the approximation error or knowing the operating frequency zone of the entire system, when constructing the approximation model, one can determine from (13) the required number of differential equations of the system (the number of concentrated masses).

The obtained approximation model has a number of unique properties that have been found useful in constructing structural models of electromechanical systems: reversibility (allows input influences and receive feedback at any point on a linearly extended object); dualism of parameters (input quantities can act as results); heterogeneity (various dependencies with respect to the spatial coordinate can be specified); multi-status (the ability to reproduce many states of an object, for example, changing boundary conditions); accessibility of parameters (the distribution of intermediate parameters by spatial coordinate is available, which, as a rule, are physical quantities, which makes it possible to evaluate, control, and control them); the possibility of operational clarification.

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**Author Information:**

**Anatoliy Feodorovich Verlan** (Institute of Modeling in Power Engineering, National Academy of Sciences of Ukraine, Chief Researcher)

**Miraziz Varisovich Sagatov** (Tashkent State Technical University, Head of the Department of "Information Technology")

**Dilbar Karimovna Karimov** (Tashkent State Technical University, Assistant Professor of the Department of "Information Technology")

**Ubaydulla Sagdullaevich Fayzullaev** (Tashkent State Technical University, senior teacher of the department "Information Technology") - informtgtu@mail.ru tel: +998977801014